

Spanning Properties of Theta-Theta Graphs^{*}

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Abstract. We study the spanning properties of Theta-Theta graphs. Similar in spirit with the Yao-Yao graphs, Theta-Theta graphs partition the space around each vertex into a set of k cones, for some fixed integer $k > 1$, and select at most one edge per cone. The difference is in the way edges are selected. Yao-Yao graphs select an edge of minimum length, whereas Theta-Theta graphs select an edge of minimum orthogonal projection onto the cone bisector. It has been established that the Yao-Yao graphs with parameter $k = 6k'$ have spanning ratio 11.67, for $k' \geq 6$. In this paper we establish a first spanning ratio of 7.82 for Theta-Theta graphs, for the same values of k . We also extend the class of Theta-Theta spanners with parameter $6k'$, and establish a spanning ratio of 16.76 for $k' \geq 5$. We surmise that these stronger results are mainly due to a tighter analysis in this paper, rather than Theta-Theta being superior to Yao-Yao as a spanner. We also show that the spanning ratio of Theta-Theta graphs decreases to 4.64 as k' increases to 8. These are the first results on the spanning properties of Theta-Theta graphs.

Keywords: Yao graph, Theta graph, Yao-Yao, Theta-Theta, spanner

1 Introduction

Let S be a set of n points in the plane, and let G be an undirected plane graph with vertex set S . The *length* of a path in G is the sum of the Euclidean lengths of its constituent edges. The distance in G between any two points $a, b \in S$ is the length of a shortest path between a and b . We say that G is a *spanner* if it preserves distances between each pair of points in S , up to a given factor. Specifically, for a fixed integer $t \geq 1$, we say that G is a *t-spanner* if any two points $a, b \in S$ at distance $|ab|$ in the plane are at distance at most $t \cdot |ab|$ in G . The smallest integer t for which this property holds is called the *spanning ratio* of G . Clearly there is a tradeoff between the spanning ratio and the sparsity of G : the smaller the spanning ratio, the denser the spanner and the better the approximation of the original distances.

One way to control the tradeoff between the spanning ratio and the sparsity of the spanner is to partition the space around each point into equiangular cones

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of angle $\theta = 2\pi/k$, for some integer $k \geq 1$, and connect each point to a “nearest” point in each cone. Intuitively, this construction promises a short detour between any two points $a, b \in S$, by following the edge from a aiming in the direction of b (the one lying in the cone with apex a containing b). The definition of a “nearest” point comes in two flavors, in the context of *Yao* graphs [31] and *Theta*-graphs (or Θ -graphs) [14,22]. For Yao graphs, the “nearest” point is simply the point that minimizes the L_2 -distance, whereas for Theta graphs, the “nearest” point in a cone C is the point whose orthogonal projection onto the bisector of C minimizes the L_2 -distance. Both Yao and Theta graphs are parameterized by a positive integer $k \geq 1$, which controls the cone angle $\theta = 2\pi/k$. In the following we will refer to the Yao graph as Y_k and Theta graphs as Θ_k , for a fixed $k \geq 1$. Both Y_k and Θ_k are known to be efficient spanners, for $k \geq 6$. The spanning ratios of these graphs are summarized in Table 1.

Parameter k	Spanning Ratio			
	Y_k	Θ_k	YY_k	$\Theta\Theta_k$
< 4	∞ [25]			
4	696.1 [9]	237 [3]	∞ [17]	
5	3.74 [2]	9.96 [11]	OPEN	∞ [23]
6	5.8 [2]	2 [4]	∞ [25]	OPEN
$k > 6$	$\frac{1}{1-2\sin(\theta/2)}$ [8]	$\frac{1}{1-\sin(\theta/2)}$ [27]	11.67 for $k = 6k'$ and $k' \geq 6$ [16]	16.76 for $k = 6k'$ and $k' \geq 5$ [HERE]
$4k + 2$		$1 + 2\sin(\theta/2)$		
$4k + 4$		$1 + \frac{2\sin(\theta/2)}{\cos(\theta/2) - \sin(\theta/2)}$ [12,7]		
$4k + 3, 4k + 5$	$\frac{1}{1-2\sin(3\theta/8)}$ [2]	$\frac{\cos(\theta/4)}{\cos(\theta/2) - \sin(3\theta/4)}$		

Table 1. Spanning ratios of Yao and Theta graphs for various $\theta = 2\pi/k$ values.

Interest in Yao and Theta graphs has increased with the advancement of wireless ad hoc networks and the need for efficient communication (see [26,28,21,15] and the references therein). Designing routing algorithms for wireless ad hoc networks is an extremely difficult task and research in this area is still in progress. The overlay communication graph formed by the wireless links should be a spanner to ensure fast delivery of information, and should also have low degree to ensure a low maintenance cost and reduced MAC-level contention and interference [20]. We observe that both Yao and Theta graphs obey the first requirement (as detailed in Table 1), but fail to satisfy the second requirement. One simple example consists of $n - 1$ points equally distributed around a circle centered at an n^{th} point p . Then, for $k \geq 6$, both Θ_k and Y_k will have an edge directed from each of the $n - 1$ points towards p , because p is “nearest” in one of their cones. So each of Θ_k and Y_k has out-degree k , but in-degree $n - 1$. To reduce the in-degree, alternate spanner structures based on Yao and Theta graphs have been proposed, such as Yao-Yao [30], Sink [24,1], Stable Roommates [6], and Ordered-Yao [29].

The *Yao-Yao* graph with integer parameter $k \geq 1$, denoted YY_k , is a subgraph of Y_k obtained by applying a second Yao step to the set of incoming edges in each cone. More precisely, for each point p and each cone with apex a containing two or more incoming edges, YY_k retains only a shortest incoming edge and discards the rest. Ties are broken arbitrarily. This construction guarantees a degree of at most $2k$ at each node in YY_k (one incoming and one outgoing edge per cone), however the spanning property of YY_k is still under investigation. The only existing result shows that $YY_{6k'}$, for $k' \geq 6$, is a spanner with spanning ratio 11.67. For $k' \geq 8$, the spanning ratio of $YY_{6k'}$ drops to 4.75 [16].

Sink spanners [24,1] transform bounded outdegree spanners, such as Y_k and Θ_k , into bounded degree spanners, by replacing each directed star consisting of all links directed into a point p and lying in a cone with apex p , by a tree of bounded degree with “sink” p . The result is a spanner with degree at most $k(k+2)$ and spanning ratio $1/(1 - 2\sin(\theta/2))^2$.

The *Stable Roommates* spanner introduced in [6] has degree at most k and spanning ratio matching the spanning ratio of Y_k , so this spanner combines both qualities – low spanning ratio and low degree – of the Yao and Yao-Yao graphs, respectively. The only drawback of this approach is that it processes pairs of points in non-decreasing order by their distances, making it unsuitable for a fast local implementation. (The authors present a distributed implementation that requires $O(n)$ rounds of communication.)

The *ordered* Theta approach [10] reduces the potentially linear degree of the Theta graph to a logarithmic degree. Similar to the stable roommates approach, the ordered Theta approach imposes a particular ordering on the input points. The authors show that careful orderings can produce graphs with spanning ratio $1/(\cos \theta - \sin \theta)$ and degree $O(k \log n)$.

Similar in spirit with the Yao-Yao graph, in this paper we introduce the *Theta-Theta* graph $\Theta\Theta_k$, parameterized by integer $k \geq 1$, and study the spanning properties of this graph. The graph $\Theta\Theta_k$ is obtained by applying a filtering step to the edges of Θ_k as follows. For each point p and each cone C with apex p , we consider all edges in C directed into p , and maintain only a “shortest” edge while discarding the rest. Recall that in the context of Theta graphs, a “shortest” edge minimizes the length of its projection on the cone bisector. Ties are arbitrarily broken.

Our main result shows that $\Theta\Theta_{6k'}$ is a spanner, for any $k' \geq 5$. This result relies on a result by Bonichon et al. [4], who prove that Θ_6 is a 2-spanner. Our main contribution is showing that $\Theta\Theta_{6k'}$ contains a short path between the endpoints of each edge in Θ_6 . More precisely, we show that for each edge $ab \in \Theta_6$, there is a path between a and b in $\Theta\Theta_{6k'}$ no longer than $8.38|ab|$, for $k' \geq 5$. This, combined with the fact that Θ_6 is a 2-spanner, yields an upper bound of 16.76 on the spanning ratio of $\Theta\Theta_{6k'}$. A similar approach has been used in [16] to establish that $YY_{6k'}$ has spanning ratio 11.67, for $k' \geq 6$. We observe that the spanning ratio of $\Theta\Theta_{6k'}$ decreases to 7.82, 5.63 and 4.64 as k' increases to 6, 7, and above 8, respectively. The spanning ratios established in this paper for $\Theta\Theta_{6k'}$ are stronger than the ones obtained in [16] for $YY_{6k'}$, for

the same parameter values $k' \geq 6$. We surmise that this is mainly due to the tighter analysis in this paper, rather than $\Theta\Theta_{6k'}$ being superior to $YY_{6k'}$ as a spanner.

1.1 Definitions

Throughout the paper, S will refer to a fixed set of n points in the plane. The directed Yao graph Y_k with integer parameter $k \geq 1$ on S is constructed as follows. For each point $a \in S$, starting with the direction of the positive x -axis, extend k equally spaced rays r_1, r_2, \dots, r_k originating at a , in counterclockwise order (see Figure 1a for $k = 6$). These rays divide the plane into k cones, denoted by $C_{\{k,1\}}(a), C_{\{k,2\}}(a), \dots, C_{\{k,k\}}(a)$, each of angle $\theta = 2\pi/k$. To avoid overlapping boundaries, we assume that each cone is half-open and half-closed, meaning that $C_{\{k,i\}}(a)$ includes r_i but excludes r_{i+1} (here $r_{k+1} \equiv r_1$ wraps around). In each cone of a , draw a directed edge from a to its “closest” point b in that cone (the one that minimizes the L_2 -distance $|ab|$). Ties are broken arbitrarily. These directed edges collectively form the edge set for the directed Yao graph. The undirected Yao graph (or simply Yao graph) on S is obtained by simply ignoring the directions of these edges. The Theta graph Θ_k is defined in a similar way, with the only difference being in the definition of “closest”: in each cone C with apex a , draw a directed edge from a to the point b that minimizes the distance between a and the orthogonal projection of b on the bisector of the cone. For example, looking at the cone $C_{\{6,1\}}(a)$ in Figure 1b, notice that b_1

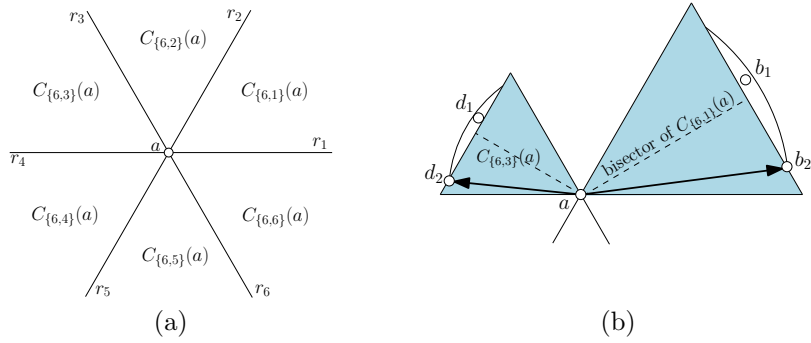


Fig. 1. Definitions (a) Rays defining the cones at point a (b) Theta edges ab_2, ad_2 .

minimizes the L_2 -distance to a , whereas b_2 minimizes the L_2 -distance between its projection onto the cone bisector and a . Consequently, $\overrightarrow{ab_1}$ will be added to Y_6 , and $\overrightarrow{ab_2}$ to Θ_6 . Similarly, $\overrightarrow{ad_1} \in C_{\{6,3\}}(a)$ will be added to Y_6 , and $\overrightarrow{ad_2}$ to Θ_6 . Figure 2a shows the Yao graph Y_6 for the point set depicted in Figure 1b, and Figure 2c shows the Theta graph Θ_6 for the same point set.

The Yao-Yao graph $YY_k \subseteq Y_k$ is obtained from Y_k by applying a reverse Yao step to the set of incoming Yao edges in Y_k . That is, for each node a and each

cone with apex a containing two or more incoming edges, YY_k retains a shortest incoming edge and discards the rest. Ties are broken arbitrarily. The Theta-Theta graph $\Theta\Theta_k \subseteq \Theta_k$ is obtained from Θ_k in a similar way, with the only difference being in the requirement that a “shortest” incoming edge in a cone minimizes the length of its projection onto the cone bisector. Figure 2b shows the graph YY_6 derived from the graph Y_6 depicted in Figure 2a, and Figure 2d shows the graph $\Theta\Theta_6$ derived from the graph Θ_6 depicted in Figure 2c.

When the choice of a particular cone is either irrelevant or is clear from the context, we ignore the cone subscript and use $C_k(a)$ to denote any of the cones $C_{\{k,1\}}(a), C_{\{k,2\}}(a), \dots, C_{\{k,k\}}(a)$. For any two points $a, b \in S$, let $C_k(a, b)$ denote the cone with apex a that contains b . Let $\triangle_k(a, b)$ be the canonical triangle with two of its sides along the rays bounding $C_k(a, b)$, and the third side orthogonal to the bisector of $C_k(a, b)$ and passing through b . For example, shaded in Figure 1b are the canonical triangles $\triangle_6(a, b_2)$ and $\triangle_6(a, d_2)$.

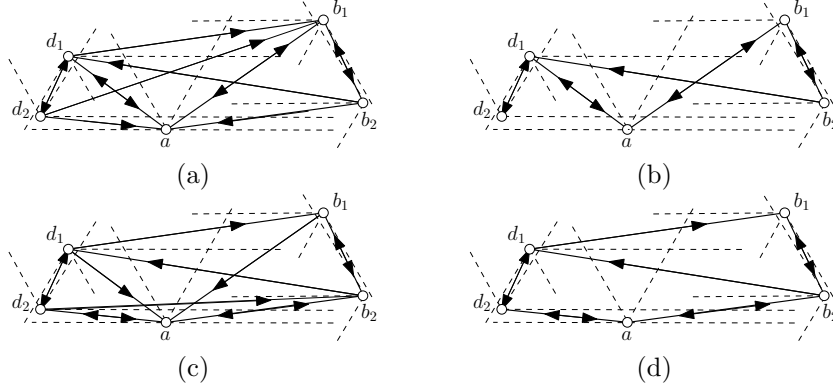


Fig. 2. Graph examples (a) Y_6 (b) YY_6 (c) Θ_6 (d) $\Theta\Theta_6$.

For any pair of vertices a and b in an undirected graph G , let $\xi_G(a, b)$ denote a *shortest* path in G between a and b . For example, $\xi_{\Theta_6}(a, b)$ refers to a shortest path in Θ_6 from a to b .

Our main goal is to establish a short path in $\Theta\Theta_k$ between the endpoints of each edge in Θ_6 . Our arguments will rely on the assumption that, for each point $a \in S$, each cone $C_k(a)$ is entirely contained in $C_6(a)$, hence $k = 6k'$. Throughout the rest of the paper, we will work with a quadruple of distinct points $a, b, b', a' \in S$ in the following configuration: \overrightarrow{ab} is an arbitrary edge in Θ_6 ; $\overrightarrow{ab'}$ is the edge in Θ_k that lies in the cone $C_k(a, b) \subset C_6(a, b)$; and $\overrightarrow{a'b'}$ is the edge in $\Theta\Theta_k$ that lies in the cone $C_k(b', a) \subset C_6(b', a)$. We will refer to this configuration as a *canonical Θ -configuration*, to avoid repeating these definitions in different contexts. For a snapshot of a canonical Θ -configuration, see ahead to Figure 4a. We will further assume, without loss of generality, that in a canonical Θ -configuration \overrightarrow{ab} lies in $C_{\{6,1\}}(a)$, and the bisector of $C_k(a, b)$

lies below, or aligns with, the bisector of $\triangle_6(a, b)$. Any other configuration is equivalent to this canonical Θ -configuration under rotational and/or reflectional symmetry.

2 Preliminaries

In this section we present a few isolated lemmas that will be used in our main proof from Section 3. For the sake of clarity and continuity in the flow of our exposition, we defer the proofs of most of these lemmas to the appendix. We encourage the reader to skip ahead to Section 3, and refer back to these lemmas from the context of Theorem 2, where their role will become evident. We begin this section with the statement of an existing result.

Theorem 1. [4] *For any pair of points $a, b \in S$, there is a path in Θ_6 whose total length is bounded above by $2|ab|$.*

The key ingredient in the result of Theorem 1 is a specific subgraph of Θ_6 , called *half- Θ_6* . This graph preserves half of the edges in Θ_6 , those belonging to non-consecutive cones. Bonichon et al. [4] show that half- Θ_6 is a *triangular-distance*¹ Delaunay triangulation, computed as the dual of the Voronoi diagram based on the triangular distance function. Combined with Chew's proof that any triangular-distance Delaunay triangulation is a 2-spanner [13], this result settles Theorem 1. The structure of Θ_6 , viewed as the union of two planar 2-spanners, has been used in establishing spanning properties of other graphs as well [5,18,16].

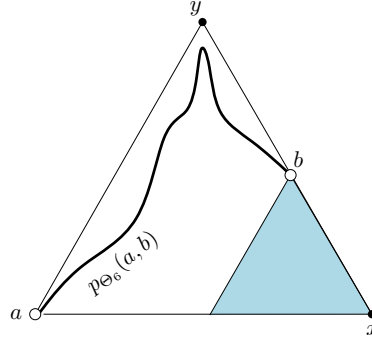


Fig. 3. Lemma 1: $|\xi_{\Theta_6}(a, b)| \leq |ay| + |by|$.

¹ The *triangular distance* from a point a to a point b is the side length of the smallest equilateral triangle centered at a that touches b and has one horizontal side.

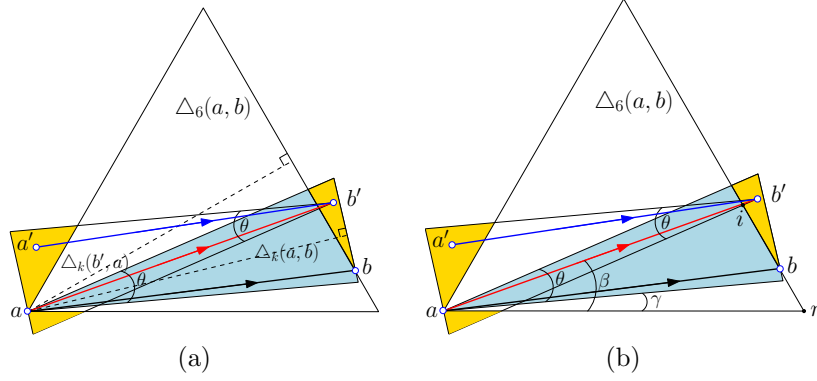


Fig. 4. (a) Canonical Θ -configuration: $ab \in \Theta_6$, $ab' \in \Theta_k$ and $a'b' \in \Theta_{\Theta_k}$ (b) Bounding $|ab'|$, $|a'b'|$ and $|bb'|$.

Before stating our preliminary results, we define the term $T(\alpha)$ parameterized by angle $\alpha \in [0, \pi/3]$ as

$$T(\alpha) = \frac{\sin(\pi/3 - \alpha) - \sin \alpha}{\sin(\pi/3)} \leq 1 \quad (1)$$

This term will occur frequently in our analysis, and this definition will come in handy. The upper bound of 1 follows from the fact that $T(\alpha)$ decreases as α increases, therefore $T(\alpha) \leq T(0) = 1$. The following lemma plays a central role in the proofs of Lemmas 3 and 4.

Lemma 1. [16] *Let $a, b \in S$ and let x and y be the other two vertices of $\Delta_6(a, b)$. If $\Delta_6(b, x)$ is empty of points in S , then $|\xi_{\Theta_6}(a, b)| \leq |ay| + |by|$. Moreover, each edge of $\xi_{\Theta_6}(a, b)$ is no longer than $|ay|$. [Refer to Figure 3.]*

Note that Lemma 1 does not specify which of the two sides ax and ay lies clockwise from $\Delta_6(a, b)$, so the lemma applies in both situations. The following lemma establishes fundamental relationships on the distances between points in a canonical Θ -configuration.

Lemma 2. *Let $a, b, b', a' \in S$ be points in a canonical Θ -configuration. Then each of $|ab'|$ and $|a'b'|$ is no longer than $|ab|/\cos(\theta/2)$. In addition, if β and γ are the angles formed by the horizontal through a with ab' and the lower ray of $C_k(a, b)$, respectively, and if $\beta \leq \pi/6$, then*

$$|ab'| \geq |ab| \frac{\sin(\pi/3 + \gamma)}{\sin(\pi/3 + \beta)}$$

[Refer to Figure 4b.]

Lemmas 3 through 5 isolate specific situations that will arise in the analysis of our main result. We state them independently in this section.

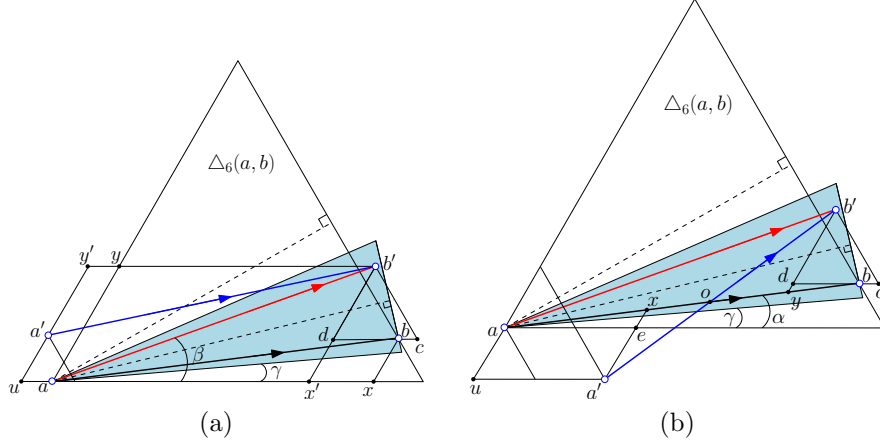


Fig. 5. Bounding $|\xi_{\Theta_6}(a, a')| + |\xi_{\Theta_6}(b, b')|$ (a) Lemma 3: a' above a (b) Lemma 4: a' below a .

Lemma 3. Let $a, b, b', a' \in S$ be points in a canonical Θ -configuration, with the additional constraint that $a' \in C_{\{6,2\}}(a)$. Let β and γ be the angles formed by the horizontal through a with ab' and the lower ray of $C_k(a, b)$, respectively. Then

$$|\xi_{\Theta_6}(a, a')| + |\xi_{\Theta_6}(b, b')| \leq (|ab| + |a'b'|) \cdot T(\gamma) - 2|ab'| \cdot T(\beta)$$

Here the term T is as defined in (1). Furthermore, each edge of $\xi_{\Theta_6}(a, a')$ and $\xi_{\Theta_6}(b, b')$ is strictly smaller than ab , for $\theta \leq \pi/6$. [Refer to Figure 5a.]

Lemma 4. Let $a, b, b', a' \in S$ be points in a canonical Θ -configuration, with the additional constraints that $a' \in C_{\{6,6\}}(a)$, and the angle α formed by ab with the horizontal through a is at most $\pi/6$. Then

$$|\xi_{\Theta_6}(a, a')| + |\xi_{\Theta_6}(b, b')| \leq |ab| - |a'b'| \cdot \frac{\sin(\pi/3 - \alpha - \theta) - \sin \theta}{\sin(\pi/3 - \alpha)}$$

Furthermore, each edge of $\xi_{\Theta_6}(a, a')$ and $\xi_{\Theta_6}(b, b')$ is strictly shorter than ab , for $\theta \leq \pi/12$. [Refer to Figure 5b.]

Lemma 5. Let $a, b, b', a' \in S$ be points in a canonical Θ -configuration, with the additional constraint that either $a' \in C_{\{6,5\}}(a)$, or $a' \in C_{\{6,6\}}(a)$ and the angle formed by ab with the horizontal through a is above $\pi/6$. Then

$$|\xi_{\Theta_6}(a, a')| + |\xi_{\Theta_6}(b, b')| \leq 8|ab| \sin(\theta/2)$$

Furthermore, each edge of $\xi_{\Theta_6}(a, a')$ and $\xi_{\Theta_6}(b, b')$ is strictly shorter than ab , for $\theta \leq \pi/15$.

Our approach to finding a short path in $\Theta\Theta_k$ between the endpoints of each edge in Θ_6 uses induction on the Euclidean lengths of the edges in Θ_6 . The following lemma will be useful in proving the inductive step in various situations.

Lemma 6. *Let $a, b, b', a' \in S$ be points in a canonical Θ -configuration, and let $t \geq 1$ be a fixed real value. Assume that, for each edge $xy \in \Theta_6$ no longer than ab , the inequality $|\xi_{\Theta\Theta_k}(x, y)| \leq t \cdot |xy|$ holds. Let $\xi_{\Theta\Theta_k}(a, b) = \xi_{\Theta\Theta_k}(a, a') \oplus a'b' \oplus \xi_{\Theta\Theta_k}(b', b)$. If $|\xi_{\Theta_6}(a, a')| < |ab|$ and $|\xi_{\Theta_6}(b', b)| < |ab|$, then*

$$|\xi_{\Theta\Theta_k}(a, b)| \leq t \cdot |\xi_{\Theta_6}(a, a')| + t \cdot |\xi_{\Theta_6}(b', b)| + |a'b'|$$

Furthermore, if $|ab| - |\xi_{\Theta_6}(a, a')| - |\xi_{\Theta_6}(b', b)| > 0$, then $|\xi_{\Theta\Theta_k}(a, b)| \leq t \cdot |ab|$ for any real value t such that

$$t \geq \frac{|ab|/\cos(\theta/2)}{|ab| - |\xi_{\Theta_6}(a, a')| - |\xi_{\Theta_6}(b', b)|}. \quad (2)$$

Here the symbol \oplus is used to denote the path concatenation operator.

Proof. Because $|\xi_{\Theta_6}(a, a')| < |ab|$, each edge on $\xi_{\Theta_6}(a, a')$ must be shorter than ab . This along with the lemma statement implies that, for each edge xy on the path $\xi_{\Theta_6}(a, a')$, the inequality $|\xi_{\Theta\Theta_k}(x, y)| \leq t \cdot |xy|$ holds. Summing up these inequalities for all edges along the path $\xi_{\Theta_6}(a, a')$ yields $|\xi_{\Theta\Theta_k}(a, a')| \leq t \cdot |\xi_{\Theta_6}(a, a')|$. Similar arguments show that $|\xi_{\Theta\Theta_k}(b', b)| \leq t \cdot |\xi_{\Theta_6}(b', b)|$. Thus the first inequality stated by this lemma holds. Using the upper bound on $|a'b'|$ from Lemma 2, and the assumption that $|ab| - |\xi_{\Theta_6}(a, a')| - |\xi_{\Theta_6}(b', b)| > 0$, this inequality can be easily reorganized into $|\xi_{\Theta\Theta_k}(a, b)| \leq t \cdot |ab|$ for any real value t that satisfies (2). \square

3 $\Theta\Theta_{6k'}$ is a Spanner, for $k' \geq 4$

This section presents our main result, which shows that $\Theta\Theta_k$ is a spanner, provided that $k = 6k'$ and $k' \geq 5$ (and so $\theta \leq \pi/15$). In particular, we show that for each edge $ab \in \Theta_6$, there is a path in $\Theta\Theta_{6k'}$ no longer than $8.38|ab|$. This, combined with the result of Theorem 1, yields our main result that $\Theta\Theta_{6k'}$ is a 16.76-spanner, for $k' \geq 5$. The spanning ratio decreases to 7.82 for $k' \geq 6$, which is superior to the spanning ratio of 11.67 established in [18] for $YY_{6k'}$, with $k' \geq 6$. We also show that the spanning ratio of $\Theta\Theta_{6k'}$ drops to 4.64 for $k' \geq 8$.

Our approach takes advantage of the fact that each edge $ab \in \Theta_6$ is embedded in an equilateral triangle $\triangle_6(a, b)$ empty of points in S . The restriction $k = 6k'$ is necessary in our analysis to guarantee that each cone used in constructing Θ_k and $\Theta\Theta_k$ is a subset of a cone used in constructing Θ_6 , therefore it inherits a large area empty of points in S . This property is crucial in establishing a “short” path in $\Theta\Theta_k$ between the endpoints of each edge in Θ_6 . Although we search for *undirected* paths in the undirected version of $\Theta\Theta_k$, we sometimes point out the direction of an edge if significant in the context.

Theorem 2. *Let $k = 6k'$ be a positive integer, with $k' \geq 5$. For each edge $\vec{ab} \in \Theta_6$, a shortest path in $\Theta\Theta_k$ between a and b satisfies $|\xi_{\Theta\Theta_k}(a, b)| \leq t \cdot |ab|$, where t is a positive real with values 8.38, 3.91, 2.811 and 2.32 corresponding to k' values 5, 6, 7, and above 8, respectively.*

Proof. Recall that $\theta = 2\pi/k$, so in the context of this theorem $\theta \leq \pi/15$. Throughout this proof will refer to the value t from the theorem statement as the *stretch* factor, with the understanding that it measures the “stretch” in $\Theta\Theta_k$ of an edge $ab \in \Theta_6$, and to be distinguished from the spanning ratio of $\Theta\Theta_k$ (which by Theorem 1 is at most $2t$).

The proof is by induction on the Euclidean length of the edges in Θ_6 . The base case corresponds to a shortest edge $\vec{ab} \in \Theta_6$. In this case we show that $\vec{ab} \in \Theta_k$ and $\vec{ab} \in \Theta\Theta_k$. Assume to the contrary that $\vec{ab} \notin \Theta_k$ and let $\vec{ab'} \in \Theta_k$ be the edge that lies in $C_k(a, b)$. Lemma 3 does not impose any restrictions on the relative position of the b and b' , therefore the result that each edge on $\xi_{\Theta_6}(b, b')$ is strictly shorter than ab applies in this context. This contradicts our assumption that ab is a shortest edge in Θ_6 . This shows that $\vec{ab} \in \Theta_k$. Similar arguments, used in conjunction with Lemmas 3, 4 and 5 (which distinguish between different locations of a' relative to a), show that $\vec{ab} \in \Theta\Theta_k$.

Our inductive hypothesis states that the theorem holds for all edges in Θ_6 of length strictly lower than some fixed value $\delta > 0$. To prove the inductive step, pick a shortest edge $\vec{ab} \in \Theta_6$ of length δ or higher, and find a “short” path $\xi_{\Theta\Theta_k}(a, b)$ that satisfies the conditions of the theorem. Let a' and b' be the other two points in S which, along with a and b , complete a canonical Θ -configuration: $\vec{ab'} \in \Theta_k$ lies in $C_k(a, b)$, and $\vec{a'b'} \in \Theta\Theta_k$ lies in $C_k(b', a)$. Refer to Figure 4a. Also recall our general assumptions that in a canonical Θ -configuration $ab \in C_{\{6,1\}}(a)$, and the bisector of $C_k(a, b)$ aligns with, or lies below, the bisector of $\Delta_6(a, b)$. The locus of b' is $\Delta_k(a, b) \setminus \Delta_6(a, b)$, which is an area completely inside $C_{\{6,1\}}(a)$. The locus of a' is $\Delta_k(b', a) \setminus \Delta_6(a, b)$, which is an area that may overlap two or three of the cones $C_{\{6,2\}}(a)$, $C_{\{6,5\}}(a)$ and $C_{\{6,6\}}(a)$. Note that a' may not lie in $C_{\{6,3\}}(a)$, due to our assumption that the bisector of $\Delta_k(a, b)$ is no higher than the bisector of $\Delta_6(a, b)$.

Our intent is to use the result of Lemma 6 to establish the existence of a path between a and b of length at most $t \cdot |ab|$, for some fixed real constant $t > 1$. The two key ingredients needed by Lemma 6 are “short” paths in Θ_6 between a and a' , and between b and b' . We discuss three cases, depending on whether a' lies in $C_{\{6,2\}}(a)$, $C_{\{6,5\}}(a)$ or $C_{\{6,6\}}(a)$. The case $a' \in C_{\{6,5\}}(a)$ is the simplest, so we will save it for last. Let α , β and γ be the angles formed by the horizontal through a with ab , ab' , and the lower ray of $C_k(a, b)$, respectively.

Case $a' \in C_{\{6,2\}}(a)$. This case is depicted in Figure 5a. By Lemma 3, we have

$$|\xi_{\Theta_6}(a, a')| + |\xi_{\Theta_6}(b, b')| \leq (|ab| + |a'b'|) \cdot T(\gamma) - 2|ab'| \cdot T(\beta) \quad (3)$$

where T is as defined in (1). Notice the restrictions on the angles β and γ :

$$\begin{aligned} 0 &\leq \gamma \leq \pi/6 - \theta/2 \\ \gamma &\leq \beta \leq \gamma + \theta \end{aligned} \quad (4)$$

The upper bound on γ is due to our assumption that the bisector of $\triangle_k(a, b)$ is no higher than the bisector of $\triangle_6(a, b)$. The bounds on β follow immediately from the definitions of γ and β . Next we determine a maximum for the quantity on the right hand side of (3). We consider two situations, depending on ranges of β , which affect the sign of $T(\beta)$. Observe that $T(\gamma)$ is always positive, since $\pi/3 - \gamma > \gamma$ for any $\gamma < \pi/6$.

Assume first that $\beta \leq \pi/6$, so ab' is no higher than the bisector of $\triangle_6(a, b)$. In this case $\beta \leq \pi/3 - \beta$ and $\sin \beta \leq \sin(\pi/3 - \beta)$, therefore $T(\beta)$ is positive. Substituting in (3) the upper bound on $|a'b'|$ and the lower bound on $|ab'|$ from Lemma 2 yields

$$\frac{|\xi_{\Theta_6}(a, a')| + |\xi_{\Theta_6}(b, b')|}{|ab|} \leq T(\gamma) + \frac{T(\gamma)}{\cos(\theta/2)} - 2T(\beta) \cdot \frac{\sin(\pi/3 + \gamma)}{\sin(\pi/3 + \beta)}$$

Let $X(\theta, \gamma, \beta)$ denote the quantity on the right hand side of the inequality above. Note that $X(\theta, \gamma, \beta)$ increases as θ increases, therefore $X(\theta, \gamma, \beta) \leq X(\pi/15, \gamma, \beta)$ for $\theta \leq \pi/15$. Figure 6a shows how $X(\theta, \gamma, \beta)$ varies with $\gamma \in [0, \pi/6 - \theta/2]$ and $\beta \in (\gamma, \min\{\pi/6, \gamma + \theta\}]$, for fixed $\theta = \pi/15$. It can be verified that $|\xi_{\Theta_6}(a, a')| +$

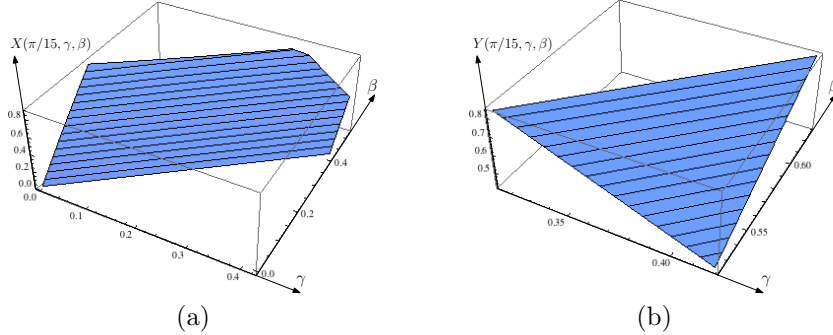


Fig. 6. Case $a' \in C_{\{6,2\}}$ (a): upper bound on $|\xi_{\Theta_6}(a, a')| + |\xi_{\Theta_6}(b, b')|$ for $\theta = \pi/15$ and $\gamma \in [0, \pi/6 - \theta/2]$ (a) $\alpha \in (\gamma, \min\{\pi/6, \gamma + \theta\}]$ (b) $\alpha \in (\pi/6, \min\{\pi/6 + \theta/2, \gamma + \theta\}]$

$|\xi_{\Theta_6}(b, b')| < 0.88|ab|$, for any $0 < \theta \leq \pi/15$. This along with Lemma 6 yields a stretch factor $t = 8.3760$ for the path in $\Theta\Theta_k$ between a and b . The stretch factor t decreases with θ as shown in the second column of Table 2.

Assume now that $\pi/6 < \beta \leq \pi/6 + \theta/2$, so ab' lies above the bisector of $\triangle_k(a, b)$. In this case $T(\beta)$ is negative, and by (4) we have $\gamma \geq \pi/6 - \theta$. Substituting in (3) the upper bound on $|ab'|$ and $|a'b'|$ from Lemma 2 yields

$$\frac{|\xi_{\Theta_6}(a, a')| + |\xi_{\Theta_6}(b, b')|}{|ab|} \leq T(\gamma) + \frac{T(\gamma) - 2T(\beta)}{\cos(\theta/2)} \quad (5)$$

θ	Case $a' \in C_{\{6,2\}}(a)$: stretch factor t from Lemma 6	
	$0 < \beta \leq \pi/6$	$\pi/6 < \beta \leq \pi/6 + \theta/2$
$\pi/15$	8.3760	6.2720
$\pi/18$	3.9058	3.3377
$\pi/21$	2.8109	2.5014
$\pi/24$	2.3159	2.1057

Table 2. Case $a' \in C_{\{6,2\}}(a)$: real constant t from Lemma 6 for various θ values.

Let $Y(\theta, \gamma, \beta)$ denote the quantity on the right hand side of the inequality above. Because $T(\gamma)$ is positive and $T(\beta)$ is negative, $T(\gamma) - 2T(\beta)$ is positive and therefore $Y(\theta, \gamma, \beta)$ increases as θ increases. It follows that $Y(\theta, \gamma, \beta) \leq Y(\pi/15, \gamma, \beta)$ for $\theta \leq \pi/15$. Figure 6b shows how $Y(\theta, \gamma, \beta)$ varies with $\gamma \in [\pi/6 - \theta, \pi/6 - \theta/2]$ and $\beta \in (\pi/6, \gamma + \theta]$, for $\theta = \pi/15$. It can be verified that $|\xi_{\Theta_6}(a, a')| + |\xi_{\Theta_6}(b, b')| < 0.8397|ab|$, for any $0 < \theta \leq \pi/15$. This along with Lemma 6 yields a stretch factor $t = 6.2720$ for the path in $\Theta\Theta_k$ between a and b . The stretch factor t decreases with θ as shown in the third column of Table 2.

Case $a' \in C_{\{6,6\}}$. This case is depicted in Figure 5b. We discuss two situations, depending on whether ab lies above or below the bisector of $\triangle_6(a, b)$. Assume first that ab is no higher than the bisector of $\triangle_6(a, b)$, so $\alpha \leq \pi/6$. Thus we are in the context of Lemma 4, which gives us an upper bound $|\xi_{\Theta_6}(a, a')| + |\xi_{\Theta_6}(b, b')| \leq |ab| - |a'b'| \cdot Z(\theta, \alpha)$, where

$$Z(\theta, \alpha) = \frac{\sin(\pi/3 - \theta - \alpha) - \sin \theta}{\sin(\pi/3 - \alpha)}$$

Note that $Z(\theta, \alpha)$ decreases as θ increases, therefore $Z(\theta, \alpha) \geq Z(\pi/15, \alpha)$ for any $\theta \leq \pi/15$. It can be verified that $Z(\theta, \alpha) \geq m = 0.2022$, for any $\theta \leq \pi/15$. By Lemma 6, we have

$$\xi_{\Theta\Theta_k}(a, b) \leq t|ab| - t|a'b'| \cdot Z(\theta, \alpha) + |a'b'|$$

Simple calculations show that the right hand side of the inequality above does not exceed $t|ab|$ for any $t \geq 4.945 \geq 1/m$. This bound decreases with θ as shown in the second column of Table 3.

Assume now that ab lies above the bisector of $\triangle_6(a, b)$, so $\alpha > \pi/6$. Intuitively, this forces a and a' to lie close to each other (for sufficiently small θ values), and similarly for b and b' , so we can work with somewhat looser upper bounds without exceeding the spanning ratio established so far. Our context matches the context of Lemma 5, which tells us that $|\xi_{\Theta_6}(a, a')| + |\xi_{\Theta_6}(b, b')| \leq X(\theta) = 8|ab|\sin(\theta/2)$. The bound $X(\theta)$ increases with θ , therefore $X(\theta) \leq X(\pi/15) \leq 0.8363$. This together with Lemma 6 yields $|\xi_{\Theta\Theta_k}(a, b)| < t \cdot |ab|$ for any $t \geq 6.1397$. This bound decreases with θ as shown in the third column of Table 3.

θ	Case $a' \in C_{\{6,6\}}(a)$: stretch factor t from Lemma 6	
	$0 \leq \alpha \leq \pi/6$	$\pi/6 < \alpha \leq \pi/6 + \theta/2$
$\pi/15$	4.9454	6.1397
$\pi/18$	2.9697	3.3157
$\pi/21$	2.3117	2.4936
$\pi/24$	1.9829	2.1020

Table 3. Case $a' \in C_{\{6,6\}}(a)$: real constant t from Lemma 6 for various θ values.

Case $a' \in C_{\{6,5\}}$. The bound on $|\xi_{\Theta_6}(a, a')| + |\xi_{\Theta_6}(b, b')|$ provided by Lemma 5 applies here as well, therefore the analysis for this case is identical to the one for the previous case (with $b' \in C_{\{6,6\}}$ and ab above the bisector of $\triangle_6(a, b)$), yielding the spanning ratios listed in the third column of Table 3.

To derive the results listed in Tables 2 and 3, we worked with a quadruplet of *distinct* points a, b, a', b' in a Θ -configuration. The cases where a and a' coincide, or b and b' coincide, are special instances of this general case and yield lower stretch factors. The results listed in Tables 2 and 3 indicate that the stretch factor is highest when a' lies above a and ab' is below the bisector of $\triangle_6(a, b)$. The largest stretch factor value is $t = 8.376$ for $\theta = \pi/15$, and it drops to 3.91, 2.82 and 2.32 for θ values $\pi/18$, $\pi/21$ and $\pi/24$, respectively. This completes the proof. \square

Combined with the result of Theorem 1, the result of Theorem 2 yields the main result of this paper, stated by Theorem 3 below.

Theorem 3. *The graph $\Theta\Theta_k$, with $k = 6k'$ and $k' \geq 5$, is a 16.76-spanner. The spanning ratio decreases to 7.82, 5.63 and 4.64 as k' increases to 6, 7, and above 8, respectively.*

4 Conclusions

In this paper we present the first results on the spanning property of $\Theta\Theta_k$ -graphs. We show that, for any integer $k' \geq 5$, the graph $\Theta\Theta_{6k'}$ is a spanner with spanning ratio 16.76. The spanning ratio drops to 7.82 for $k' \geq 6$, which is superior to the spanning ratio of 11.67 established in [18] for $YY_{6k'}$, with $k' \geq 6$. The framework of our analysis seems inadequate to handle all graphs $\Theta\Theta_k$, for all $k > 6$, because it relies on the fact that each cone used in constructing $\Theta\Theta_k$ is a subset of a cone used in constructing Θ_6 . It is unclear whether a fundamentally new technique is required to handle all $\Theta\Theta_k$ graphs, for $k \geq 6$. Proving or disproving that these graphs are spanners remains the main open problem in this area.

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Appendix: Deferred Proofs

4.1 Proof of Lemma 2

Lemma 2. *Let $a, b, b', a' \in S$ be points in a canonical Θ -configuration. Then each of $|ab'|$ and $|a'b'|$ is no longer than $|ab|/\cos(\theta/2)$. In addition, if β and γ are the angles formed by the horizontal through a with ab' and the lower ray of $C_k(a, b)$, respectively, and if $\beta \leq \pi/6$, then*

$$|ab'| \geq |ab| \frac{\sin(\pi/3 + \gamma)}{\sin(\pi/3 + \beta)}$$

[Refer to Figure 4b.]

Proof. Let h be the height of the isosceles triangle $\triangle_k(a, b)$, and let s be the length of its two equal sides. They are related by $h = s \cos(\theta/2)$. Because both ab and ab' lie inside $\triangle_k(a, b)$, their length may not exceed s . Also $|ab|$ may not be lower than h , since b is on the base of $\triangle_k(a, b)$. This implies $|ab| \geq h = s \cos \theta/2 \geq |ab'| \cos \theta/2$, so the upper bound on $|ab'|$ holds. Observe now that $\triangle_k(a, b)$ and $\triangle_k(b', a)$ are similar, and the side length of $\triangle_k(b', a)$ does not exceed s (because b' lies inside $\triangle_k(a, b)$). Similar arguments can then be used to establish the same upper bound on $|a'b'|$.

Let i be the intersection point between ab' and the right side of $\triangle_6(a, b)$. Then $|ab'| \geq |ai|$. By the Law of Sines applied on $\triangle abi$, we have $|ab|/\sin \angle aib = |ai|/\sin \angle abi$. Let r be the lower right corner of $\triangle_6(a, b)$. Note that $\angle aib = \pi/3 - \beta$ (as angle interior to $\triangle air$) and $\angle abi = \pi/3 + \angle bar \geq \pi/3 + \gamma$ (as angle exterior to $\triangle abr$). Also because $\beta \leq \pi/6$, $\angle abi$ is acute, therefore $\sin \angle abi \geq \sin(\pi/3 + \gamma)$. These together show that $|ai| \geq |ab| \sin(\pi/3 + \gamma)/\sin(\pi/3 + \alpha)$, so the lower bound on $|ab'|$ holds. This completes the proof. \square

4.2 Proof of Lemma 3

Lemma 3. *Let $a, b, b', a' \in S$ be points in a canonical Θ -configuration, with the additional constraint that $a' \in C_{\{6,2\}}(a)$. Let β and γ be the angles formed by the horizontal through a with ab' and the lower ray of $C_k(a, b)$, respectively. Then*

$$|\xi_{\Theta_6}(a, a')| + |\xi_{\Theta_6}(b, b')| \leq (|ab| + |a'b'|) \cdot T(\gamma) - 2|ab'| \cdot T(\beta)$$

Here the term T is as defined in (1). Furthermore, each edge of $\xi_{\Theta_6}(a, a')$ and $\xi_{\Theta_6}(b, b')$ is strictly smaller than ab , for $\theta \leq \pi/6$. [Refer to Figure 5a.]

Proof. First we determine an upper bound on $|\xi_{\Theta_6}(b, b')|$. Let c and d be the right and left corners of $\triangle_6(b', b)$, respectively. Because b' is interior to $\triangle_k(a, b)$, the perpendicular from b' to the bisector of $\triangle_k(a, b)$ intersects the line segment ab , so the perpendicular from b' to cd falls left of b . This implies that $|bc| \leq |bd|$. Note that $\triangle_6(b', b)$ meets the conditions of Lemma 1, with $\triangle_6(b, d)$ empty of points in S , therefore $|\xi_{\Theta_6}(b, b')| \leq |bc| + |b'c| \leq |bd| + |b'd|$. Let the horizontal

through a intersect the left rays of $C_{\{6,5\}}(b)$ and $C_{\{6,5\}}(b')$ in points x and x' , respectively. Then $|bd| = |ax| - |ax'|$ and $|b'd| = |b'x'| - |bx|$, so we have

$$|\xi_{\Theta_6}(b, b')| \leq (|ax| - |ax'|) + (|b'x'| - |bx|) \quad (6)$$

We determine $|ax|$ and $|bx|$ in terms of $|ab|$ by applying the Law of Sines on $\triangle abx$: $|ax|/\sin \angle abx = |bx|/\sin \angle bax = |ab|/\sin \angle axb$. Note that $\angle axb = 2\pi/3$, therefore both $\angle bax$ and $\angle abx$ are acute. This along with the fact that $\angle bax \geq \gamma$ implies $\sin \angle bax \geq \sin \gamma$, and $\angle abx = \pi/3 - \angle bax \leq \pi/3 - \gamma$ implies $\sin \angle abx \leq \sin(\pi/3 - \gamma)$. Combining these inequalities together yields

$$|ax| \leq |ab| \cdot \frac{\sin(\pi/3 - \gamma)}{\sin(\pi/3)} \quad \text{and} \quad |bx| \geq |ab| \cdot \frac{\sin \gamma}{\sin(\pi/3)} \quad (7)$$

Next we determine $|b'x'|$ and $|ax'|$ in terms of $|ab'|$ by applying the Law of Sines on $\triangle ab'x'$: $|ax'|/\sin \angle ab'x' = |b'x'|/\sin \angle b'ax' = |ab'|/\sin(\pi/3)$. Plugging in the angle values $\angle b'ax' = \beta$ and $\angle ab'x' = \pi/3 - \beta$ yields

$$|ax'| = |ab'| \cdot \frac{\sin(\pi/3 - \beta)}{\sin(\pi/3)} \quad \text{and} \quad |b'x'| = |ab'| \cdot \frac{\sin \beta}{\sin(\pi/3)} \quad (8)$$

Combining inequalities (6), (7) and (8) together yields

$$|\xi_{\Theta_6}(b, b')| \leq |ab| \cdot T(\gamma) - |ab'| \cdot T(\beta) \quad (9)$$

Next we determine an upper bound on $|\xi_{\Theta_6}(a, a')|$. Let u be the left corner of $\triangle_6(a', a)$ (refer to Figure 5a.) By Lemma 1, $|\xi_{\Theta_6}(a', a)| \leq |au| + |a'u|$. Let the horizontal through b' intersect the left side of $\triangle_6(a, b)$ and the line supporting $a'u$ in points y and y' , respectively. Then $|au| = |b'y'| - |b'y|$ and $|a'u| = |ay| - |a'y'|$. These together imply

$$|\xi_{\Theta_6}(a, a')| \leq (|ay| - |a'y'|) + (|b'y'| - |b'y|) \quad (10)$$

Note that $|ay| = |b'x'|$ and $|b'y| = |ax'|$, so the bounds from (8) apply here as well. Next we determine $|a'y'|$ and $|b'y'|$ in terms of $|a'b'|$ by applying the Law of Sines on $\triangle a'b'y'$: $|a'y'|/\sin \angle a'b'y' = |b'y'|/\sin \angle y'a'b' = |a'b'|/\sin \angle a'y'b'$. Because the upper ray of $C_k(b', a')$ is parallel to the lower ray of $C_k(a, b)$, we have $\angle a'b'y' \geq \gamma$ and $\angle y'a'b' = \pi/3 - \angle a'b'y' \leq \pi/3 - \gamma$. Since both angles are acute, we get $\sin \angle a'b'y' \geq \sin \gamma$ and $\sin \angle y'a'b' \leq \sin(\pi/3 - \gamma)$. These together imply

$$|a'y'| \geq |a'b'| \cdot \frac{\sin \gamma}{\sin(\pi/3)} \quad \text{and} \quad |b'y'| \leq |a'b'| \cdot \frac{\sin(\pi/3 - \gamma)}{\sin(\pi/3)} \quad (11)$$

Combining inequalities (10), (8) and (11) together yields

$$|\xi_{\Theta_6}(a, a')| \leq |a'b'| \cdot T(\gamma) - |ab'| \cdot T(\beta)$$

This along with (9) settles the first part of the lemma. We now turn to the second claim of the lemma. By Lemma 1, each edge on $\xi_{\Theta_6}(b, b')$ is no longer than $|b'd| \leq$

$|b'x'| = |ab'| \sin \beta / \sin(\pi/3)$ (cf. (8)), and each edge on $\xi_{\Theta_6}(a, a')$ is no longer than $|a'u| \leq |ay| = |b'x'|$. To simplify discussion, let $A = |ab'| \sin \beta / \sin(\pi/3)$. It suffices to show that $A < |ab|$ in order to settle the second part of the lemma. Because the bisector of $\triangle_k(a, b)$ is no higher than the bisector of $\triangle_6(a, b)$, we have that $\beta \leq \pi/6 + \theta/2$, therefore $A < |ab'| \sin(\pi/6 + \theta/2) / \sin(\pi/3)$. Substituting the upper bound on $|ab'|$ from Lemma 2 yields

$$A < |ab| \frac{\sin(\pi/6 + \theta/2)}{\sin(\pi/3) \cos(\theta/2)}$$

It can be verified that the right hand side of this inequality is strictly smaller than $|ab|$, for any $\theta \leq \pi/6$. This completes the proof. \square

4.3 Proof of Lemma 4

Lemma 4. *Let $a, b, b', a' \in S$ be points in a canonical Θ -configuration, with the additional constraints that $a' \in C_{\{6,6\}}(a)$, and the angle α formed by ab with the horizontal through a is at most $\pi/6$. Then*

$$|\xi_{\Theta_6}(a, a')| + |\xi_{\Theta_6}(b, b')| \leq |ab| - |a'b'| \cdot \frac{\sin(\pi/3 - \alpha - \theta) - \sin \theta}{\sin(\pi/3 - \alpha)}$$

Furthermore, each edge of $\xi_{\Theta_6}(a, a')$ and $\xi_{\Theta_6}(b, b')$ is strictly shorter than ab , for $\theta \leq \pi/12$. [Refer to Figure 5b.]

Proof. We define the following points: c and d are the right and left corners of $\triangle_6(b', b)$; u is the left corner of $\triangle_6(a', a)$; x and e are the points where the right ray of $C_{\{6,2\}}(a')$ intersects ab and the horizontal through a , respectively; y is the point where the line supporting $b'd$ intersects ab ; and o is the intersection point between ab and $a'b'$. Refer to Figure 5b. Arguments similar to the ones used in the proof of Lemma 3 show that $|\xi_{\Theta_6}(b, b')| \leq |b'd| + |bd|$. This along with $|b'd| \leq |b'y|$ and $|bd| \leq |by|$ implies

$$|\xi_{\Theta_6}(b, b')| \leq |by| + |b'y|$$

By Lemma 1 we have $|\xi_{\Theta_6}(a, a')| \leq |a'u| + |au| = |ae| + |a'e| \leq |ax| + |a'x|$. This together with the inequality above and the fact that $|by| + |ax| = |ab| - |xy|$, yields

$$|\xi_{\Theta_6}(a, a')| + |\xi_{\Theta_6}(b, b')| \leq |ab| - |xy| + (|a'x| + |b'y|) \quad (12)$$

Using the similarity property of $\triangle a'ox$ and $\triangle b'oy$, we derive $|xy| = |a'b'| \cdot |xo|/|a'o|$ and $|a'x| + |b'y| = |a'b'| \cdot |a'x|/|a'o|$. Using the Law of Sines on $\triangle a'ox$, we derive $|xo|/|a'o| = \sin \angle xa'o / \sin \angle a'xo$ and $|a'x|/|a'o| = \sin \angle a'ox / \sin \angle a'xo$. Observe that $\angle a'ox \leq \theta$ (because the ray shooting from b' towards a , parallel to ab , lies inside $C_k(b', a')$ of angle θ , and $\angle a'ox$ is equal to the angle formed by this ray with $a'b'$), and $\angle a'xo = 2\pi/3 + \alpha$ (as angle exterior to $\triangle aex$). It follows that $\angle xa'o > \pi/3 - \alpha - \theta$. These together imply

$$|xy| > |a'b'| \cdot \frac{\sin(\pi/3 - \alpha - \theta)}{\sin(\pi/3 - \alpha)} \quad \text{and} \quad |a'x| + |b'y| < |a'b'| \cdot \frac{\sin \theta}{\sin(\pi/3 - \alpha)}$$

These inequalities along with (12) yield the upper bound on $|\xi_{\Theta_6}(a, a')| + |\xi_{\Theta_6}(b, b')|$ stated by this lemma.

For the second part of the lemma, it can be verified that the term $(\sin(\pi/3 - \alpha - \theta) - \sin \theta) / \sin(\pi/3 - \alpha)$ is strictly positive for any $t \in (0, \pi/12]$ and $\alpha \in [0, \pi/3]$. This along with the upper bound established by this lemma shows that $|\xi_{\Theta_6}(a, a')| + |\xi_{\Theta_6}(b, b')| < |ab|$, therefore each edge on each of the paths $\xi_{\Theta_6}(a, a')$ and $\xi_{\Theta_6}(b, b')$ is strictly shorter than ab . This completes the proof. \square

4.4 Proof of Lemma 5

Lemma 5. *Let $a, b, b', a' \in S$ be points in a canonical Θ -configuration, with the additional constraint that either $a' \in C_{\{6,5\}}(a)$, or $a' \in C_{\{6,6\}}(a)$ and the angle formed by ab with the horizontal through a is above $\pi/6$. Then*

$$|\xi_{\Theta_6}(a, a')| + |\xi_{\Theta_6}(b, b')| \leq 8|ab| \sin(\theta/2)$$

Furthermore, each edge of $\xi_{\Theta_6}(a, a')$ and $\xi_{\Theta_6}(b, b')$ is strictly shorter than ab , for $\theta \leq \pi/15$.

Proof. The conditions stated by the lemma suggest that either a and a' lie close to each other (if $a' \in C_{\{6,5\}}(a)$), or b and b' lie close to each other (if ab is above the bisector of $\triangle_6(a, b)$). Intuitively, the upper bounds established for these two cases must be within a small factor of each other.

Let o be the intersection point between ab and $a'b'$. By the lemma statement b' lies below the horizontal through a , therefore the point o exists. Observe that a ray shooting from b' towards a , parallel to ab , lies inside $C_k(b', a')$ of angle θ , and $\angle a oa'$ is equal to the angle formed by this ray with $a'b'$, therefore $\angle a oa' \leq \theta$. By the Law of Sines applied on triangle $\triangle a oa'$, we have $|aa'| / \sin \angle a oa' = |oa'| / \sin \angle a' ao$. This along with Theorem 1 and the fact that $\angle a oa' \leq \theta$ implies

$$|\xi_{\Theta_6}(a, a')| \leq 2|oa'| \cdot \frac{\sin \theta}{\sin \angle a' ao} \quad (13)$$

Similarly arguments used on $\triangle bob'$ show that

$$|\xi_{\Theta_6}(b, b')| \leq 2|ob'| \cdot \frac{\sin \theta}{\sin \angle b' bo} \quad (14)$$

Consider first the case where $a' \in C_{\{6,6\}}(a)$, and ab is above the bisector of $\triangle_6(a, b)$. In this case $\angle b' bo > \pi/2$ and $\angle a' ao > \pi/6$ (since a' is below the horizontal through a). By the definition of a Θ -configuration, the bisector of $C_k(a, b)$ lies below the bisector of $\triangle_6(a, b)$, therefore the angle formed by ab with the bisector of $\triangle_6(a, b)$ is at most $\theta/2$. It follows that $\angle a' ao < \pi/2 + \theta/2 < \pi/2 + \pi/6$ and similarly $\angle b' bo < \pi/2 + \pi/6$. These together show that $\sin \angle a' ao > \sin(\pi/6)$ and $\sin \angle b' bo > \sin(\pi/6)$, which along with (13) and (14) yield

$$|\xi_{\Theta_6}(a, a')| + |\xi_{\Theta_6}(b, b')| \leq 2|a'b'| \cdot \frac{\sin \theta}{\sin(\pi/6)} = 4|a'b'| \sin \theta \quad (15)$$

Substituting the upper bound on $|a'b'|$ from Lemma 2 results in $|\xi_{\Theta_6}(a, a')| + |\xi_{\Theta_6}(b, b')| \leq 4|ab| \sin \theta / \cos(\theta/2) = 8|ab| \sin(\theta/2)$. Thus the upper bound claimed by the lemma holds for this case.

Assume now that $b' \in C_{\{6,5\}}(a)$. In this case $\angle a'ao \geq \pi/3$, and similarly $\angle b'bo \geq \pi/3$ (because b' lies exterior to $\triangle_6(a, b)$ and above b). Since neither of these angles can extend as far as $\pi/2 + \pi/3$, the inequalities $\sin \angle a'ao \geq \sin(\pi/3)$ and $\sin \angle b'bo \geq \sin(\pi/3)$ hold. These along with (13) and (14) yield

$$|\xi_{\Theta_6}(a, a')| + |\xi_{\Theta_6}(b, b')| \leq 2|a'b'| \cdot \frac{\sin \theta}{\sin(\pi/3)} < 2|a'b'| \cdot \frac{\sin \theta}{\sin(\pi/6)}$$

This shows that the bound from (15) established for the previous case applies in this case as well. This settles the first part of the lemma. For the second part, simple calculations show that $8 \sin(\theta/2) < 1$ for any $\theta \leq \pi/15$. This implies that $|\xi_{\Theta_6}(a, a')| + |\xi_{\Theta_6}(b, b')| < |ab|$, therefore each edge of $\xi_{\Theta_6}(a, a')$ and $\xi_{\Theta_6}(b, b')$ is strictly shorter than ab . This completes the proof. \square